

# Improving bounds on packing densities of 4-point permutations

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## Abstract

We consolidate what is currently known about packing densities of 4-point permutations and in the process improve current lower bounds for the packing densities of 1324 and 1342. We also provide rigorous upper bounds for the packing densities of 1324, 1342, and 2413. All our bounds are within  $10^{-4}$  of the true packing densities. Together with the known bounds, this gives us a fairly complete picture of all 4-point packing densities. Our main tool for the upper bounds is the framework of flag algebras introduced by Razborov in 2007.

## 1 Introduction

In this paper, we study packing densities of small permutations. A *permutation* is an ordered tuple utilizing all integers from  $\{1, \dots, n\}$ . We say that  $S = S[1]S[2] \dots S[m]$  is a *sub-permutation* of  $P = P[1]P[2] \dots P[n]$  if there exists an  $m$ -subset  $\{k_1, \dots, k_m\}$  of  $\{1, \dots, n\}$  such that for all  $1 \leq i, j \leq m$ ,  $S[i] < S[j]$  whenever  $P[k_i] < P[k_j]$ . We denote by  $N(S, P)$  the number of occurrences of  $S$  as a sub-permutation of  $P$ . Let  $\mathcal{P}_n$  be the class of all permutations of length  $n$ . If  $N(S, n) = \max_{P \in \mathcal{P}_n} N(S, P)$ , then the *packing density* of  $S$  is defined to be  $p(S) = \lim_{n \rightarrow \infty} N(S, n) / \binom{n}{m}$ .

S	lower bound	ref LB	upper bound	ref UB
1234	1	trivial	1	trivial
1432	$\beta \approx 0.423570$	[Pri97]	$\beta \approx 0.42357$	[Pri97]
2143	3/8	trivial	3/8	[Pri97]
1243	3/8	trivial	3/8	[AAH <sup>+</sup> 02]
1324	<b>0.244</b>	[Pri97]	<b>X</b>	[Pri97]
1342	$\gamma \approx \mathbf{0.196579}$	[Bat]	<b>0.1988373</b>	[BHL <sup>+</sup> 15]
2413	$\approx 0.104724$	[PS10]	<b>0.1047805</b>	[BHL <sup>+</sup> 15]

Table 1: Overview of packing densities for 4-point permutations. For 1432,  $\beta = 6\sqrt[3]{\sqrt{2}-1} - 6/\sqrt[3]{\sqrt{2}-1+4}$ . For 1342,  $\gamma = (2\sqrt{3}-3)\beta$ . The X means that we “know” the packing density is close to 0.244 but there is no upper bound. The items in bold are updated by the current work.

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The study of permutation packing densities began after Wilf’s 1992 SIAM address. Galvin (unpublished) soon rediscovered Katona, Nemetz, and Simonovits’ [KNS64] averaging argument from graph theory, and proved that  $p(S)$  exists for all  $S$ . In 1993, Stromquist, and independently Galvin and Kleitman (both unpublished), found the packing density of 132. Up to symmetry, 132 is the only permutation of length 3 with non-trivial packing density. See Table 1 alongside the rest of this paragraph. Price [Pri97] determined packing densities of numerous *layered patterns*. In 2002, Albert, Atkinson, Handley, Holton, and Stromquist [AAH<sup>+</sup>02] proved a tight upper bound for 1243, and upper bounds of  $2/9$  for both 2413 and 1342. In 2010, Presutti and Stromquist [PS10] gave the current lower bound for the packing density of 2413. In 2015, Balogh, Hu, Lidický, Pikhurko, Udvari, and Volec [BHL<sup>+</sup>15] mention that they were able to obtain upper bounds of 0.1047805 and 0.1988373 for 2413 and 1342, respectively. They do not discuss them any further.

It is worthwhile to point out that Balogh et al. [BHL<sup>+</sup>15] used flag algebras to attack the packing density problem for monotone sequences of length 4. To the best of our knowledge, the only other application of flag algebras to permutations packing, although indirect, is by Falgas-Ravry and Vaughan [FRV13]. In 2013, they obtained the inducibility (as packing density is referred to in graph theory) of a 2-star directed graph  $\bullet \rightarrow \bullet \leftarrow \bullet$ . Their result implies the known upper bound for the packing density of 132. Later, Huang [Hua14] used an argument exploiting equivalence classes of vertices to extend the result to all directed  $k$ -stars. This argument was known in the permutations setting since Price [Pri97] used it to establish the packing densities  $p(1k\dots 2)$  for all  $k$ . Similarly, flagmatic was available since 2013, yet it was not used to obtain an upper bound on the packing density of 1324. Hence, we decided to collect, enhance, and improve results in permutation packing densities that utilize flag algebras.

The rest of this note is structured as follows. The aim of section 2 is to introduce notation and concepts, including the part of flag algebras that we need. While Razborov [Raz07] presented flag algebras in the general setting of a universal model theory without constants and function symbols, we choose permutations to be the structures on which we base our exposition. We hope this makes the presentation clearer. Section 3 presents the main results of this chapter. We use flag algebras to provide upper bounds for the packing densities of 4-point permutations 1324, 1342, and 2413. We learned belatedly about the existence of the latter two bounds from [BHL<sup>+</sup>15]. Regarding lower bounds, we give a new lower-bound construction for the packing density of 1342 that almost meets the upper bound. In case of 1324, we provide a lower bound that agrees with the upper bound on the first five decimal places.

In addition to the mathematical content, we make available a flag algebras package for permutations, **Permpack**, written as a **Sage** [Dev17] script. It does all our computations and can be used for further research.

## 2 Definitions and concepts

A *pattern* of length  $k$ , where  $k \leq n$ , is a  $k$ -tuple of integers from  $[n] := \{1, \dots, n\}$ . Pattern of length  $n$  is called a *permutation*. We write tuples as strings: 1324 stands for  $(1, 3, 2, 4)$ . Two patterns  $P$  and  $S$  are *identical*, if  $P[i] = S[i]$  for all  $i \in [k]$ . They are *order-isomorphic* if for all pairs of indices  $i, j$ , it holds that  $P[i] < P[j]$  implies  $S[i] < S[j]$ . For a set  $I = \{i_1, \dots, i_m\}$  of  $m$  indices from  $[n]$ , the *sub-pattern*  $P[I]$  is the  $m$ -tuple  $P[i_1]P[i_2] \cdots P[i_m]$ . By overloading

the notation slightly, we also use  $P[I]$  to refer to the *sub-permutation* of length  $m$  that is order-isomorphic to the sub-pattern  $P[I]$ .

A *decreasing (increasing) permutation* of length  $k$  is the  $k$ -tuple  $k \dots 321$  ( $123 \dots k$ ). A permutation  $P$  is *layered*, if it is an increasing sequence of decreasing permutations. To be exact, a layered permutation can be partitioned into “sub-tuples”  $P = P_1 P_2 \dots P_\ell$  such that for all  $1 \leq i \leq \ell$ ,  $P_i$  is a decreasing sequence of consecutive integers satisfying the following: if  $x \in P_i$  and  $y \in P_j$  with  $i < j$ , then  $x < y$ . For instance, 321465987 can be partitioned as 321|4|65|987, so it is layered.

Given  $S$  and  $P$  of lengths  $m$  and  $n$ , respectively, we let  $N(S, P)$  denote the number of times that  $S$  occurs as a subpermutation of  $P$ . Then the *density* of  $S$  in  $P$  is

$$p(S, P) = \frac{N(S, P)}{\binom{n}{m}}.$$

If  $n < m$ , we set  $p(S, P) = 0$ . Intuitively,  $p(S, P)$  is the probability that a random  $m$ -set of positions from  $[n]$  induces a pattern in  $P$  that is order-isomorphic to  $S$ . For example,  $p(12, 132) = 2/3$  as both 13 and 12 are order-isomorphic to 12 while 32 is not. We want our classes of permutations to be closed under taking subpermutations. With this requirement, we can still consider permutation classes avoiding a set  $\mathcal{F}$  of *forbidden* permutations. We say that  $P$  is  $\mathcal{F}$ -free if  $N(S, P) = 0$  for all  $S \in \mathcal{F}$ . Such  $P$  is also said to *avoid*  $\mathcal{F}$  or be *admissible*. We denote by  $\mathcal{P}_n$  the set of all *admissible* permutations of length  $n$ .



Figure 1: Permutations in  $\mathcal{P}_3$ . From left to right: 123, 132, 213, 312, 231, 321.

Let  $P \in \mathcal{P}_n$  and  $S \in \mathcal{P}_m$  with  $m \leq n$ . The maximum value of  $p(S, P)$  over  $P \in \mathcal{P}_n$  is denoted by  $p(S, n)$ . Conversely, a permutation  $P$  such that  $p(S, P) = p(S, n)$  is called an  $S$ -*maximiser* of length  $n$ . It is well-known that for every  $S$ , the sequence  $(p(S, n))_{n \geq 0}$  converges to a value in  $[0, 1]$  because it is non-increasing and stays between 0 and 1. See [KNS64]. This allows us to define the packing density.

**Definition 1.** Let  $S$  be a fixed permutation and  $\mathcal{P} = \cup_{n \geq 1} \mathcal{P}_n$  the set of admissible permutations. The *packing density* of  $S$  is

$$p(S) = \lim_{n \rightarrow \infty} p(S, n).$$

For example, the packing density of 12 in 123-free permutations is  $1/2$ , because the maximiser of size  $n$  has at most two layers and by optimizing their sizes it follows that they are balanced, i.e.  $\lfloor n/2 \rfloor$  and  $\lceil n/2 \rceil$ .

Let  $(P_n)_n = P_1, P_2, P_3, \dots$  be a sequence of permutations of increasing lengths. We say that  $(P_n)_n$  is *convergent* if for every permutation  $S$ ,  $(p(S, P_n))_{n=1}^\infty$  converges. A *permuton*  $\mu$  is a probability measure with uniform marginals on the Borel  $\sigma$ -algebra  $\mathcal{B}([0, 1]^2)$ , i.e. for every  $a, b \in [0, 1]$  with  $a < b$ , it holds that  $\mu([a, b] \times [0, 1]) = b - a = \mu([0, 1] \times [a, b])$ . See examples of permutons in Figure 2.

Let  $\mu$  be a permuton and  $S$  a permutation on  $[m]$ . One can sample  $m$  points from  $[0, 1]^2$  according to  $\mu$  and with probability one they will be in general position (no two aligned vertically or horizontally). We define  $p(S, \mu)$  as the probability that a randomly sampled

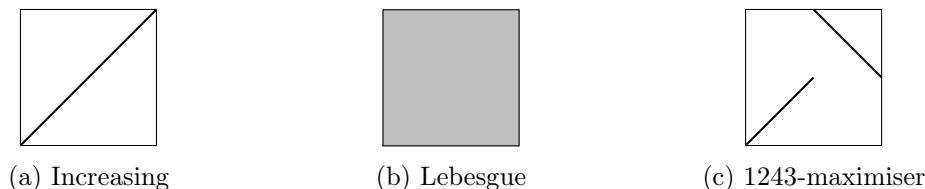


Figure 2: Examples of permutons. In (a) we have the limit of  $(1 \dots n)_{n=1}^\infty$ , in (b) it is, with probability one, the limit of a sequence of randomly chosen permutations of each length, and in (c) we have the limit of  $(1 \dots \lfloor n/2 \rfloor n \dots \lceil n/2 \rceil)_{n=1}^\infty$ .

$m$  points from  $[0, 1]^2$  according to  $\mu$  are order-isomorphic to  $S$ . It turns out that every convergent sequence of permutations has its permuton and vice versa. In particular, Hoppen, Kohayakawa, Moreira, Ráth, and Sampaio [HKM<sup>+</sup>13] proved that for every  $(P_n)_{n \geq 0}$  there exists a unique permuton  $\mu$  such that for every  $S$ ,  $p(S, \mu) = \lim_{n \rightarrow \infty} p(S, P_n)$ . In this sense,  $\mu$  is the limit of the sequence  $(P_n)_n$ . In the other direction, they proved that if  $\mu$  is a permuton and  $P_n$  is a permutation of length  $n$  sampled at random according to  $\mu$  from  $[0, 1]^2$ , then with probability one the sequence  $(P_n)_n$  is convergent (with  $\mu$  as its limit). The concept of permutation limits was known as “packing measures” since Presutti and Stromquist [PS10] used them for constructing the 2413 lower bound. In the current work, we also use permutons mainly to describe extremal constructions that yield our lower bounds.

## 2.1 Flag Algebras

Flag algebras is a framework first introduced by Razborov [Raz07]. It proved to be a very useful tool for researchers in extremal graph theory, but was used in other fields as well. For a fairly comprehensive overview of flag algebras aided results, see Razborov’s own survey [Raz13]. It is important to note that the method of flag algebras (or FA method) has evolved from combinatorial and analytic methods in combinatorics that had been used by researchers for a long time. The Cauchy-Schwarz type arguments were exploited by e.g. Bondy [Bon97] as early as 1990s, the ideas pertinent to quasirandom graphs have been around since Chung, Graham, and Wilson [CGW88]. And while there are other analytic methods that were used successfully to attack extremal problems in combinatorics, the FA method is syntactical and allows for automation. This is also the main feature that distinguishes the theory of flag algebras from the theory of dense graph limits (see e.g. Lovász [Lov12]).

Our structure of choice is permutations although the FA method works for many other combinatorial objects (including graphs, hypergraphs, partial orders). This section describes the method in as much depth as we need but no further. For more extensive expositions, see the PhD theses of Sperfeld [Spe12] and Volec [Vol14]. By now, there are also many papers with explanations and examples such as [BT11], [FRMPV15], [FRV12], [FRV13]. For a long list of important results across disciplines of discrete mathematics that were aided by flag algebras, see the abovementioned theses, especially Chapter 1 of [Vol14]. The main flag algebra result in permutations is [BHL<sup>+</sup>15]. Král’ and Pikhurko [KP13] also mention flag algebraic approach in their work on quasirandom permutations.

Consider the following example. Assume we are looking for  $\bullet \bullet$ -free permutations  $P$  that maximise the density of  $\bullet \bullet$ . We get the following bound without much effort.

$$\begin{aligned}
p(\bullet, \bullet, P) &= \underbrace{p(\bullet, \bullet, 123)p(123, P) + p(\bullet, \bullet, 132)p(132, P) + p(\bullet, \bullet, 213)p(213, P)}_{=0} \\
&\quad + p(\bullet, \bullet, 231)p(231, P) + p(\bullet, \bullet, 312)p(312, P) + p(\bullet, \bullet, 321)p(321, P) \\
&= \max\{2/3, 2/3, 1/3, 1/3, 0\}
\end{aligned}$$

However, a simple observation is that there is no  $P$  of length more than 4 such that  $p(132, P) + p(213, P) = 1$ . This is because by Erdős-Szekeres theorem, a length 5 permutation contains either a 123 or 321. So there are always 3-sets in  $P$  which do not induce 132 nor 213 and the bound  $2/3$  cannot be achieved. Knowing this, it would be useful to be able to consider how copies of 132 and 213 interact in  $P$ . The FA method helps us address this systematically by considering ways in which subpermutations can overlap inside a larger structure. This helps to shift the weight (by optimization) away from the big  $p(\bullet, \bullet, P)$  coefficients.

In general, the process is analogous when packing any permutation  $S$ . Having picked some reasonably small  $N$ , the crude bound looks as follows.

$$\begin{aligned}
p(S, P) &= \sum_{P' \in \mathcal{P}_N} p(S, P')p(P', P) \\
&\leq \max_{P' \in \mathcal{P}_N} p(S, P')
\end{aligned} \tag{1}$$

**Definition 2** (Flag). A permutation  $\tau$ -flag  $S^\tau$  is a permutation  $S$  together with a distinguished subpermutation  $\tau$ , also called an intersection *type*.

$$\begin{array}{c|c|c|c}
\bullet & \bullet & \bullet & \bullet \\
\circ & \circ & \circ & \circ \\
S_1^1 & S_2^1 & S_3^1 & S_4^1
\end{array}$$

Figure 3: If  $\tau = 1$ , then four  $\tau$ -flags of length 2 are as above. The empty circle marks  $\tau$ .

See Figure 3 for a list of all 1-flags on two vertices. The set of all admissible  $\tau$  flags of length  $m$  is denoted by  $\mathcal{P}_m^\tau$ . If  $\tau$  is the permutation of length 0 or 1, we write  $\mathcal{P}_m^0$  and  $\mathcal{P}_m^1$ , respectively, instead of  $\mathcal{P}_m^\tau$ . Notice that  $\mathcal{P}_m^0 = \mathcal{P}_m$ . The *support*  $T$  of  $\tau$  in  $S^\tau$  is the set of indices of  $S$  that span  $\tau$  in  $S^\tau$ . We say that two permutation flags  $S_1(\tau_1)$  and  $S_2(\tau_2)$  are *type-isomorphic* if  $S_1 = S_2$  and if the supports of  $\tau_1$  and  $\tau_2$  are identical. For instance, in Figure 3,  $S_1^1$  and  $S_4^1$  are not type-isomorphic, because the support of  $\tau$  in  $S_1^1$  is 1 and in  $S_4^1$  it is 2. For convenience, we will refer to the length of  $\tau$  as  $t$ .

**Definition 3.** Let  $S^\tau$  be a  $\tau$ -flag of length  $m$ ,  $P^\tau$  a  $\tau$ -flag of length  $n \geq m$ . We define  $N(S^\tau, P^\tau)$  to be the number of  $m$ -sets  $M$  in  $P$  such that  $P[M]$  is type-isomorphic to  $S^\tau$ . Flag density is then defined as follows

$$p(S^\tau, P^\tau) = \frac{N(S^\tau, P^\tau)}{\binom{n-t}{m-t}}.$$

In other words,  $p(S^\tau, P^\tau)$  is the probability that a uniformly at random chosen  $m$ -set  $M$  from  $[n]$  containing  $\tau$  induces a permutation flag  $P[M]^\tau$  which is type-isomorphic to  $S^\tau$ . For instance, consider the following flag densities.

$$p(\bullet \circ \bullet, \bullet \circ \bullet) = 1, \quad p(\bullet \circ \bullet, \bullet \circ \bullet) = 0, \quad p(\bullet \circ \bullet, \bullet \circ \bullet) = 1/2$$

Finally, we define joint density of two flags,  $p(S_1^\tau, S_2^\tau; P^\tau)$  as the probability that choosing an  $m_1$ -set  $M_1$  (containing  $\tau$ ) from  $P$  and choosing an  $m_2$ -set  $M_2$  (containing  $\tau$ ) from  $P$  such that  $M_1 \cap M_2 = \tau$  induces  $\tau$ -flags  $M_1^\tau$  and  $M_2^\tau$  in  $P^\tau$  which are type-isomorphic to  $S_1^\tau$  and  $S_2^\tau$ , respectively. The following proposition turns out to be useful (Lemma 2.3 in [Raz07]). It says that choosing subflags with or without replacement makes no difference asymptotically.

**Proposition 1.** *Let  $S_1^\tau$  and  $S_2^\tau$  be flags on  $m_1$  and  $m_2$  vertices. Let  $n \geq m_1 + m_2 - t$  and  $P^\tau$  be a flag on  $n$  vertices. Then*

$$p(S_1^\tau, P^\tau)p(S_2^\tau, P^\tau) = p(S_1^\tau, S_2^\tau; P^\tau) + o(1),$$

where  $o(1) \rightarrow 0$  as  $n \rightarrow \infty$ .

Let  $\ell = |\mathcal{P}_m^\tau|$  and fix an order on elements of  $\mathcal{P}_m^\tau$ , let  $S_i^\tau, S_j^\tau \in \mathcal{P}_m^\tau$  and  $P \in \mathcal{P}_n^\tau$ , let  $\mathbf{x}$  be a vector with  $i$ -th entry  $p(S_i^\tau, P^\tau)$ , and let  $Q^\tau$  be a positive semi-definite matrix with dimensions  $\ell \times \ell$ . Then by Proposition 1 and since  $Q^\tau \succeq 0$ , we have

$$0 \leq \mathbf{x}Q^\tau \mathbf{x}^T = \sum_{i,j \leq \ell} Q_{ij}^\tau p(S_i^\tau, S_j^\tau; P^\tau) + o(1).$$

Moreover, if we let  $\sigma$  be a type induced by a randomly chosen  $t$  vertices in  $P \in \mathcal{P}_n$ , then we have the same inequality. Averaging over choices of  $\sigma$  preserves the non-negativity as well. We denote the new flag  $P^\sigma$  by  $(P, \sigma)$  to leave trace of its origin from randomly chosen  $\sigma$  in  $P$ .

$$0 \leq \mathbb{E}_\sigma (\mathbf{x}Q^\tau \mathbf{x}^T) = \sum_{i,j \leq \ell} Q_{ij}^\tau \frac{1}{\binom{n}{t}} \sum_{\sigma \in \binom{[n]}{t}} p(S_i^\tau, S_j^\tau; (P, \sigma)) + o(1). \quad (2)$$

Furthermore, we can write the above expression in terms of permutations on  $N$  vertices.

$$\begin{aligned} \mathbb{E}_\sigma (\mathbf{x}Q^\tau \mathbf{x}^T) &= \sum_{i,j \leq \ell} Q_{ij}^\tau \frac{1}{\binom{n}{t}} \sum_{\sigma \in \binom{[n]}{t}} \sum_{P' \in \mathcal{P}_N} p(S_i^\tau, S_j^\tau; (P', \sigma)) p(P', P) + o(1) \\ &= \sum_{P' \in \mathcal{P}_N} \underbrace{\left( \sum_{i,j \leq \ell} Q_{ij}^\tau \frac{1}{\binom{n}{t}} \sum_{\sigma \in \binom{[n]}{t}} p(S_i^\tau, S_j^\tau; (P', \sigma)) \right)}_{\alpha(P', m, \tau)} p(P', P) + o(1) \end{aligned}$$

Notice that the last expression is of the form  $\sum_{P' \in \mathcal{P}_N} \alpha(P', \tau, m) p(P', P)$ . We can take such an expression for various types  $\tau$  and various  $m$ . Every such choice will require another matrix  $Q^\tau$ . In practice, we first choose  $N$ , then take all possible pairs of  $t$  and  $m$  such that  $N = 2m - t$ . Thus once  $N$  is fixed, the choice of  $t$  determines the rest. Therefore, let  $\alpha(P') = \sum_\tau \alpha(P', m, \tau)$  and recall that the expression that we are trying to minimise, subject to  $Q^\tau \succeq 0$  for all  $\tau$ , comes from (1). By adding inequalities of the form of (2) to (1), we obtain

$$\begin{aligned} p(S, P) &= \sum_{P' \in \mathcal{P}_N} p(S, P') p(P', P) \\ &\leq \sum_{P' \in \mathcal{P}_N} p(S, P') p(P', P) + \sum_{P' \in \mathcal{P}_N} \alpha(P') p(P', P) \\ &\leq \max_{P' \in \mathcal{P}_N} \{p(S, P') + \alpha(P')\}. \end{aligned} \quad (3)$$

This problem (3) has the form of a semidefinite programming problem subject to the condition that  $Q^\tau \succeq 0$  for every type  $\tau$ . There exist numerical solvers, such as CSDP or SDPA, that we can use. However, the solution is in the form of numerical PSD matrices. These need to be converted to exact (without floating-point entries) in a way that preserves their PSD property and still yields a bound that we are satisfied with. Since none of our bounds is tight, we will take a shortcut in rounding. Let  $Q'$  be a numerical matrix returned by the solver. Since it is positive semi-definite, it admits a Cholesky decomposition into a lower and upper triangular matrices:  $Q' = L'L'^T$ . We compute this decomposition and then round the  $L'$  matrices into  $L$  matrices in such a way that they do not have negative entries on the diagonals. In certificates, we provide these  $L$  matrices instead of  $Q$  matrices. This way, one can readily check that  $Q = LL^T \succeq 0$  by inspecting the diagonal entries of the  $L$  matrices.

## 2.2 Example

We have a lower bound of  $2\sqrt{3} - 3 \approx 0.464101615\dots$  for the packing density of 132 obtained from a standard construction. Assume we want to obtain an upper bound for the packing density of 132. Let  $P$  be a (large) 132-maximiser of length  $n$  and let  $3 \leq \ell \leq n$ . By (1) we get

$$\begin{aligned} p(132) &\leq p(132, P) \\ &= \sum_{P' \in \mathcal{P}_\ell} p(132, P')p(P', P) \\ &\leq \max_{P' \in \mathcal{P}_\ell} p(132, P'). \end{aligned}$$

We choose  $\ell = 3$  and set  $\lambda = 2\sqrt{3} - 3$ . Now consider

$$\begin{aligned} \Delta &= \lambda p(123, P) + (\lambda - 1)p(132, P) + \lambda p(213, P) + \frac{5\lambda - 3}{6}p(231, P) \\ &\quad + \frac{5\lambda - 3}{6}p(312, P) + \lambda p(321, P). \end{aligned}$$

Adding  $\delta$  linear combinations of  $P'$  densities to the previous crude upper bound improves it to  $\lambda$ .

$$\begin{aligned} p(132, P) &\leq \sum_{P' \in \mathcal{P}_\ell} p(132, P')p(P', P) + \Delta \\ &\leq \max_{P' \in \mathcal{P}_\ell} \left\{ \lambda, \lambda, \lambda, \frac{5\lambda - 3}{6}, \frac{5\lambda - 3}{6}, \lambda \right\} \\ &= \lambda \end{aligned}$$

The key property of  $\Delta$  is that it is non-negative for all  $P$ , including  $P' \in \mathcal{P}_3$ . Let  $\sigma$  be a randomly chosen vertex out of the three available. The matrix  $Q$  below is positive semi-definite and  $\mathbf{x}_{P'}$  is a vector of flag densities for flags in Figure 3:

$$\mathbf{x}_{P'} = (p(\circ \bullet, (P', \sigma)) \quad p(\bullet \circ, (P', \sigma)) \quad p(\bullet \circ, (P', \sigma)) \quad p(\bullet \circ, (P', \sigma))).$$

$$Q = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \lambda & \lambda & 3(\lambda - 1)/2 \\ 0 & \lambda & \lambda & 3(\lambda - 1)/2 \\ 0 & 3(\lambda - 1)/2 & 3(\lambda - 1)/2 & 3\lambda \end{pmatrix} \quad (4)$$



Averaging over  $\sigma$  gives the expression (5) that makes the non-negativity of  $\Delta$  apparent.

$$\Delta = \mathbb{E}_{\sigma} \left( \sum_{P' \in \mathcal{P}_3} \mathbf{x}_{P'} Q \mathbf{x}_{P'}^T \right) \geq 0 \quad (5)$$

Therefore, we proved that  $p(132) \leq 2\sqrt{3} - 3$ .

## 2.3 Implementation

Flagmatic 2.0 was written by Emil R. Vaughan and is currently the only general implementation of Razborov’s flag algebra framework which is freely available to use and modify. See [Vau13] for more information. The project is hosted at <http://github.com/jsliacan/flagmatic>. Unfortunately, Flagmatic does not support permutations. For this reason, we wrote Permpack, a lightweight implementation of flag algebras on top of SageMath’s Sage 7.4 [Dev17]. It does not have all the functionality of Flagmatic but it is sufficient for basic tasks. For more information, code, and installation instructions, see <https://github.com/jsliacan/permpack>.

Let us consider an example of how Permpack can be used on the above example of 132-packing. It will be clear from Permpack’s output where the  $Q$  matrix above comes from. In Permpack, one needs to specify the complexity in terms of  $N$ , the length of the admissible permutations which all computations are expressed in terms of. The `density_pattern` argument specifies the permutation whose packing density we want to determine. Once permutations, types, flags, and flag products are computed, we can delegate the rest of the tasks to the solver of our choice (currently supported solvers are `csdp` and `sdpa_dd`). The answer is a numerical upper bound on  $p(132)$ . It can be rounded automatically to a rational bound by the `exactify()` method of the `PermProblem` class. The certificate contains admissible permutations, flags, types, matrices  $Q$  (as  $L$  matrices in  $Q$ ’s Cholesky decomposition) and the actual bound as a rational number (fraction). These are sufficient to verify the bound. Below is the script used to obtain the numerical  $Q'$  matrix for the packing density of 132 with Permpack.

Listing 1: Packing 132 with Permpack.

```
p = PermProblem(3, density_pattern="132")
p.solve_sdp()
```

Listing 2: Output.

```
...
Success: SDP solved
Primal objective value: -4.6410162e-01
Dual objective value: -4.6410162e-01
Relative primal infeasibility: 5.90e-14
Relative dual infeasibility: 1.67e-10
Real Relative Gap: 3.68e-10
XZ Relative Gap: 6.14e-10
```

It is not difficult to guess the entries of  $Q$  from the numerical matrix below, which is part of the output of the SDP solver. The resulting exact matrix  $Q$  is shown in (4).

## 3 Results

The following theorem will be needed later. There exist further variations of it, e.g. Proposition 2.1 and Theorem 2.2 in [AAH<sup>+</sup>02]. However, we only need the original version.



Listing 3: Floating point  $Q'$  matrix.

[ 4.55854035127455e-10	6.806084489120e-12	6.8060845047452e-12	-1.032045390820e-10]
[ 6.80608448912001e-12	0.4641016162301893	0.464101613919741	-0.8038475767936814]
[ 6.80608450474521e-12	0.464101613919741	0.4641016162301782	-0.8038475767936717]
[-1.03204539082084e-10	-0.803847576793681	-0.8038475767936717	1.3923048450288649]

**Theorem 1** (Stromquist [Str93]). *Let  $S$  be a layered permutation. Then for every  $n$ , extremal value of  $p(S, n)$  is achieved by a layered permutation. Moreover, if  $S$  has no layer of size 1, every maximiser of  $p(S, n)$  is layered.*

### 3.1 Packing 1324

Price [Pri97] studied packing densities of layered permutations in depth. He came up with an approximation algorithm that, at  $m$ -th iteration assumes that the extremal construction has  $m$  layers (see Theorem 1) and optimises over their sizes. The algorithm then proceeds to increase  $m$  and halts when increasing  $m$  does not improve the estimate. In that case, an optimal construction has been found (up to numerical noise from the optimization, if any). In reality, the procedure is stopped manually when approximation is fine enough or complexity too high. Therefore, for every  $m$ , the value that Price's algorithm gives is a lower-bound for the packing density in question.

It is known that the extremal construction for the packing density of 1324 is layered with infinite number of layers. See e.g. [AAH<sup>+</sup>02] and [Pri97]. The main theorem of this section is the following.

**Theorem 2.**

$$0.244054321 < p(1324) < 0.244054549$$

*Proof.* Consider the construction  $\Gamma$  from Figure 4.  $\Gamma$  is a permuton. Let  $C$  denote the middle layer of  $\Gamma$  (the largest layer),  $B$  denote the layer above (and  $B'$  the layer below)  $C$ , and  $A$  denotes the group of the remaining layers above  $B$  (and  $A'$  denotes the group of layers below  $B'$ ). So  $\Gamma = A' \oplus B' \oplus C \oplus B \oplus A$ , where  $A \oplus B$  means that the layer  $A$  is entirely below and to the left of the layer  $B$ . Let  $c = |C|$ ,  $b = |B| = |B'|$ , and  $a = |A| = |A'|$ . We assume that  $A$  (and  $A'$ ) is isomorphic to a maximiser for the packing of 132-pattern (213-pattern). The aim is to optimise over  $a$  and  $b$ . Ideally, the tails of  $\Gamma$  would also be optimised over, but that is infeasible. So we assume the tails are 132 (213) maximisers. It turns out that the first two steps give a good lower bound. We now compute the density of 1324 patterns in  $\Gamma$ . There are four distinct (i.e. up to symmetry) positions that a copy of 1324 can assume in  $\Gamma$ . Let  $xyzw$  be the four points in  $\Gamma$  that form a copy of 1324 in that order.

1.  $y, z \in C$ ,  $x \in A' \cup B'$ ,  $w \in A \cup B$ , there are  $N_1$  such copies
2.  $y, z \in B$ ,  $x \in A' \cup B' \cup C$ ,  $w \in A$ , there are  $N_2$  such copies
3.  $y, z, w \in A$ ,  $x \in A' \cup B' \cup C \cup B$ , there are  $N_3$  such copies
4.  $x, y, z, w \in A$ , there are  $N_4$  such copies

Let us now determine quantities  $N_1, \dots, N_4$ .

1.  $N_1 = c^2/2 + (a+b)^2$
2.  $N_2 = b^2/2 + a(a+b+c)$
3.  $N_3 = (2\sqrt{3}-3)\frac{a^3}{6} \cdot (a+2b+c)$
4.  $N_4 = \sum_{k=0}^{\infty} \frac{\sqrt{3} \cdot (2\sqrt{3}-3)}{6 \cdot (\sqrt{3}+1)^{4k+4}} \cdot a^4$ .

Finally, we get the density of 1324 pattern in  $\Gamma$ . Let  $b = 1 - c - 2a$ . Then

$$\begin{aligned} p(1324, \Gamma) &= \max_{\substack{0 < c \leq 1/2 \\ 0 < a \leq 1/4}} 24 \cdot (N_1 + 2N_2 + 2N_3 + 2N_4) \\ &> 0.244054321. \end{aligned}$$

This proves the lower bound in Theorem 2, because  $0.244054321 < p(1324, \Gamma) \leq \phi(1324)$ .

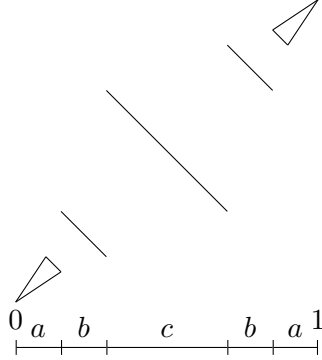



Figure 4: Construction  $\Gamma$  (a permuton). The “triangles” at the ends represent permutons that are extremal for the packing of 132 and 213.

We proved the upper bound with Flagmatic [Vau13]. Since Flagmatic does not work with permutations, and since 1324 is layered, we transformed the problem to an equivalent problem in directed graphs.

**Lemma 1.** *Let  $\mathcal{F}$  be the set of forbidden digraphs  $\{\begin{smallmatrix} \bullet & \bullet & \bullet \\ \nearrow & \rightarrow & \searrow \end{smallmatrix}, \begin{smallmatrix} \bullet & \bullet & \bullet \\ \rightarrow & \rightarrow & \rightarrow \end{smallmatrix}, \begin{smallmatrix} \bullet & \bullet & \bullet \\ \rightarrow & \rightarrow & \rightarrow \end{smallmatrix}\}$ . The packing density of 1324 equals the Turán  $\begin{smallmatrix} \bullet & \bullet & \bullet \\ \nearrow & \rightarrow & \searrow \end{smallmatrix}$ -density of  $\mathcal{F}$ . In other words,*

$$p(1324) = p(\begin{smallmatrix} \bullet & \bullet & \bullet \\ \nearrow & \rightarrow & \searrow \end{smallmatrix}, \mathcal{F}).$$

*Proof of Lemma 1.* There is a unique way to encode a layered permutation  $P$  as a directed graph  $D$ . If and only if two points  $x, y \in P$  form a 12 pattern, then  $xy$  is an arc  $x \rightarrow y$  in  $D$ . Forbidding  $\begin{smallmatrix} \bullet & \bullet & \bullet \\ \nearrow & \rightarrow & \searrow \end{smallmatrix}$ ,  $\begin{smallmatrix} \bullet & \bullet & \bullet \\ \rightarrow & \rightarrow & \rightarrow \end{smallmatrix}$ , and  $\begin{smallmatrix} \bullet & \bullet & \bullet \\ \rightarrow & \rightarrow & \rightarrow \end{smallmatrix}$  in  $D$  forces it to be a union of independent sets with arcs between them so that if  $x, y$  are vertices in one independent set and  $u, v$  are vertices in another independent set of  $D$ , then if  $xu$  is an arc in  $D$ , so are  $xv$ ,  $yu$ , and  $yv$ . In other words, all arcs between two independent sets are present, and all go in the same direction. Moreover, the direction is transitive ( $\begin{smallmatrix} \bullet & \bullet & \bullet \\ \nearrow & \rightarrow & \searrow \end{smallmatrix}$  is forbidden). Together with the first rule about the direction of arcs between independent sets, this fully characterizes the digraph  $D$  from the permutation  $P$ . Clearly, the process is reversible.  $\square$

Given Lemma 1, we use flag algebra method on directed graphs to compute an upper bound for the packing density of  over  $\{\triangleleft, \rightarrow, \rightarrow\rightarrow, \rightarrow\rightarrow\rightarrow\}$ -free digraphs. The resulting bound is the one in Theorem 2. Flagmatic script is in Listing 4. The certificate resides here: <https://github.com/jsliacan/permpack/blob/master/certificates/cert1324.js>. Note that this is a Flagmatic certificate and can be verified using the `inspect_certificate.py` script that comes with Flamgatic. Then it only remains to verify the script.

Listing 4: Packing 132

```
p = OrientedGraphProblem(8,
    forbid_induced=["3:12", "3:1223", "3:122331"],
    density="4:1213142434")
p.solve_sdp(show_output=True, solver="csdp")
p.make_exact(10^20)
```

□

## 3.2 Packing 1342

### 3.2.1 Lower bound

The previous lower bound for the packing density of 1342 was approximately 0.1965796 [Bat], [AAH<sup>+</sup>02].

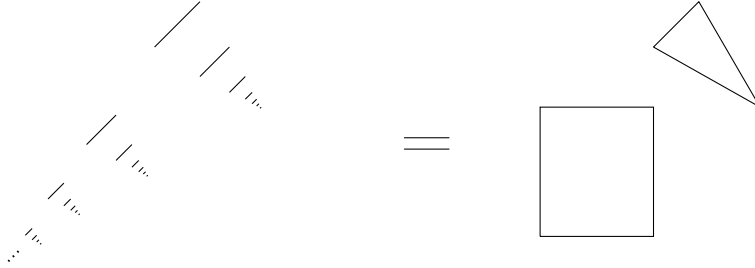


Figure 5: On the left is Batkeyev's construction [Bat] for the lower bound on  $p(1342)$  as product of packing densities of 132 and 1432. On the right is the schematic drawing of it. The triangle stands for a 231-maximiser and the square stands for the part inside which the entire construction is iterated.

Let  $\lambda = 2\sqrt{3} - 3$  be the packing density of 132 and  $\kappa$  the ratio between the ratio of the top layer to the rest of the construction in the recursive definition of 1432 maximiser, see Price [Pri97] ( $\kappa$  is the root of  $3x^4 - 4x + 1$ ). Batkeyev suggested to replace each layer in the maximiser of 1432 by a 132-maximiser while preserving the size ratio  $\kappa$ . The density of 1342 in Batkeyev's construction (see Figure 5) is

$$\begin{aligned}
 p(1342, B) &= (8\sqrt{3} - 12) \cdot \sum_{n=0}^{\infty} (1 - \kappa)^3 \kappa^{4n+1} \\
 &= p(132)p(1432) \\
 &= 2 \left( 2\sqrt{3} - 3 \right) \left( 3\sqrt[3]{\sqrt{2} - 1} - \frac{3}{\sqrt[3]{\sqrt{2} - 1}} + 2 \right) \\
 &\approx 0.1965796 \dots
 \end{aligned}$$

This lower bound was widely regarded as possibly optimal. We show a better construction below.

**Theorem 3.**

$$0.198836597 < p(1342) < 0.198837287.$$

*Proof.* The new lower bound is given by a construction  $\Pi$  in Figure 6. The weights that we used to compute the lower bound are given in full length in the file [https://github.com/jsliacan/permpack/blob/master/cert1](https://github.com/jsliacan/permpack/blob/master/certificates/cert1)

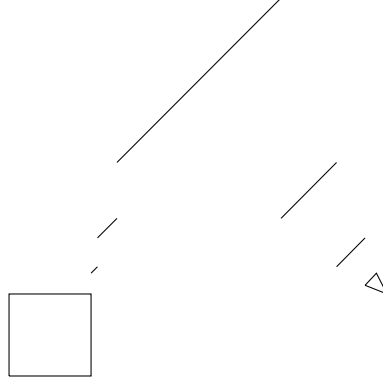


Figure 6: New lower bound construction  $\Pi$  for the packing density of 1342. The part sizes, left to right, are approximately 0.2174, 0.0170, 0.0516, 0.4341, 0.1480, 0.0764, 0.0554. The square part represents the part inside which the whole construction is applied recursively. The triangle part is the extremal construction for 231-packing.

$$\begin{aligned} a_1 &= 0.2174127723536347308692444843 \\ a_2 &= 0.0170598057899242722740620549 \\ a_3 &= 0.0516101402487892270230230972 \\ a_4 &= 0.4340722809873864994312953007 \\ a_5 &= 0.1479895625950390496250611829 \\ a_6 &= 0.0764457255805656971383351365 \\ a_7 &= 0.0554097124446605236389787433 \end{aligned}$$

Label the 7 parts of  $\Pi$  from left to right as  $a_1, \dots, a_7$ . We assign the weights to them roughly as above. Then a straightforward calculation of the 1342 density in  $\Pi$  implies the desired lower bound.

The non-FA upper bound stands at  $2/9$  [AAH<sup>+</sup>02]. The upper bound above was obtained via the FA method and confirms the claimed bound from [BHL<sup>+</sup>15]. The certificate is included with the rest of the certificates at <https://github.com/jsliacan/permpack/blob/master/certificates/cert1>. It contains the necessary information to verify the result. We used  $N = 7$  for our computations. While it is possible that  $N = 8$  would yield a slightly better bound, the computations would be much more expensive. Without a candidate for an exact lower bound, we were satisfied with the bound we obtained on  $N = 7$ .  $\square$

### 3.3 Packing 2413

The case of 2413 is fairly complicated as can be seen from the lower bound construction by Presutti and Stromquist [PS10]. The previous non-FA upper bound was  $2/9$  by Albert et

al. [AAH<sup>+</sup>02]. The bound below was obtained via flag algebras and is in the same range as the bound in [BHL<sup>+</sup>15].

**Theorem 4.**

$$p(2413) < 0.10478046354353523761779.$$

*Proof.* The FA certificate can be found with the other materials here <https://github.com/jsliacan/permpack/b>. We used admissible permutations of length  $N = 7$ . Again, larger  $N$  could yield a slightly better upper bound, but without an exact lower bound, this effort would not be justified.  $\square$

## 4 Conclusion

S	lower bound	ref LB	upper bound	ref UB
1234	1	trivial	1	trivial
1432	$\beta$	[Pri97]	$\beta$	[Pri97]
2143	3/8	trivial	3/8	[Pri97]
1243	3/8	trivial	3/8	[AAH <sup>+</sup> 02]
1324	<b>0.244054321</b>	–	<b>0.244054549</b>	–
1342	<b>0.198836597</b>	–	<b>0.198837286342</b>	–
2413	$\approx 0.104724$	[PS10]	<b>0.104780463544</b>	–

Table 2: Overview of packing densities for 4-point permutations now. For 1432,  $\beta = 6\sqrt[3]{\sqrt{2}-1} - 6/\sqrt[3]{\sqrt{2}-1+4}$ .

While we now know the packing densities of all 4-point permutations with accuracy of 0.01%, finding candidates for optimal constructions in case of 1324- and 1342-packing remains a challenge. In case of 1324, an idea for the part ratios will be needed to come up with a possible extremal construction. As for the 1342 pattern, the extremal construction might use a different layer formation than our  $\Pi$ . Even if  $\Pi$  has the right structure, the part ratios remain to be determined precisely. The latest status of 4-point packing densities is depicted in Table 2.

We must also mention an interesting line of enquiry that was made precise as Conjecture 9 in [AAH<sup>+</sup>02]. For a packing of pattern  $S$ , is there an extremal construction with infinite number of layers? Are all extremal constructions of that form? More precisely, let an  $S$ -maximiser be an  $n$ -permutation  $P$  such that  $p(S, n) = p(S, P)$ . If  $L_n$  is the number of layers in a layered maximiser of length  $n$ , what can we say about  $L_n$  as  $n \rightarrow \infty$ ? For example, we know that the number of layers in every 1324-maximiser is unbounded as  $n \rightarrow \infty$ . We also know that a 2143-maximiser has only 2 layers, regardless of  $n$ .

Finally, we would like to thank Robert Brignall for heaps of discussions and for pointing out several useful facts.

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